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AFDELING TOEGEPASTE WISKUNDE TN 99/81 JUNI
(DEPARTMENT OF APPLIED MATHEMATICS)

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ON A CLASS OF SENSITIVE CHEAP CONTROL PROBLEMS

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 49A10, 49B10, 49E25, 34D15

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ABSTRACT

For a class of linear singular optimal control problems with a non-unique singular arc, an asymptotic solution of the corresponding nearly singular problem is constructed and its validity is proved from the convergence of the power series solution of the Riccati equation.

KEY WORKS & PHRASES: *Cheap control, convergence of formal power series solutions*

*) to appear in Int. Symp. on Math. Theory of Networks and Systems vol. 4, August 5 to 7, 1981, Univ. of California, Los Angeles.

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1. INTRODUCTION

In this paper we study a class of linear time-invariant, n-dimensional dynamical systems of the type

(1ab) $\dot{x} = Ax + Bv, \quad x(0) = x_0,$

with performance index

(1c) $J = \int_0^\infty x'Qx + \varepsilon^2 v'Rv \, dt, \quad 0 < \varepsilon << 1,$

where Q is a symmetric positive semi-definite matrix and R is symmetric and positive definite. We denote the n-dimensional state space by X. The control vector takes its values in the linear m-dimensional space U and $v(\cdot)$; $\mathbb{R}^+ \to U$ is assumed to be a piece-wise continuous mapping. We will analyse the corresponding algebraic Riccati equation in the limit $\varepsilon \to 0$ for the case

(2ab) $X = K + B$ and $K \cap B \neq 0$,

where K = Ker Q and B = Im B. As we pointed out in [2], the singular optimal control problem, that is (1) with $\varepsilon = 0$, may, under certain conditions, have a family of solutions. By the conditions (2ab) we limit ourselves to a representive subclass of problems with a non-unique singular arc. In this paper we sketch the method of proving rigourously the correctness of the asymptotic approximation of (1) satisfying (2). It is assumed that the pair (A,B) is stabilizable and the pair (C,A) with C'C = Q is observable. In addition to this similar types of hypotheses holding on a subsystem of (1) will be made later on.

2. CONVERGENCE OF A FORMAL POWER SERIES

First we remark that the algebraic Riccati equation that corresponds with (1) has a unique symmetric positive definite solution, as we may conclude from the following theorem by Wonham [6].

THEOREM 1. *The class of symmetric, positive definite matrices contains a unique element that satisfies*

(3) $A'P + PA + PBR^{-1}B'P + C'C = 0$

provided that (A,B) *is stabilizable and* (C,A) *is observable.*

For proving the convergence of the formal power series solution of the algebraic Riccati equation we need the following theorem of Hautus [4].

THEOREM 2. *Let the functions* $F_{ij}(Z,\varepsilon)$, $i,j = 1,\ldots,n$ *with* $Z = \{Z_{ij}\}_{i,j=1,\ldots,n}$ *be analytic in a neighborhood of* $(Z,\varepsilon) = (0,0)$ *and let* $Z = \hat{Z}(\varepsilon)$ *be a formal power series solution of the equation* $F(Z;\varepsilon) = 0$. *If* def $F_Z(\hat{Z}(\varepsilon);\varepsilon) \neq 0$ *in a neighborhood of* $\varepsilon = 0$, *then* $\hat{Z}(\varepsilon)$ *is convergent.*

Thus, if for the algebraic Riccati equation we can prove that def $F_Z(\hat{Z}(\varepsilon);\varepsilon) \neq 0$ and that the symmetric power series expansion for the solution is positive definite, then it represents a symmetric positive definite analytic function satisfying the Riccati equation. According to Theorem 1 this function is the unique gain function that corresponds with the optimal solution.

3. THE OPTIMAL SOLUTION FOR $\varepsilon \to 0$

The system (1) can be transformed into

$$\begin{pmatrix} \dot{x}_s \\ x_k \end{pmatrix} = \begin{pmatrix} A_s & 0 \\ A_{k_s} & A_k \end{pmatrix} \begin{pmatrix} x_s \\ x_k \end{pmatrix} + \begin{pmatrix} B_s & 0 \\ 0 & B_k \end{pmatrix} \begin{pmatrix} u_s \\ u_k \end{pmatrix},$$

$$\begin{pmatrix} x_s(0) \\ x_k(0) \end{pmatrix} = \begin{pmatrix} x_{s0} \\ x_{k0} \end{pmatrix}$$

with performance index

$$\text{(4c)} \qquad J = \int_0^\infty x_s' Q_s x_s + \varepsilon^2 \{x_k' M_k x_k + 2x_k' N_{ks} U_s + 2x_k' N_k u_k + u_s' R_s u_s + 2u_k' R_{ks} u_s + u_k' R_k u_k\}\, dt,$$

where for $x = (x_s, x_k)$ such a n-dimensional basis is taken that $(0,x_k) \in K$ and $(x_s,0) \notin K$.

It is known that (4) has an optimal solution with

$$\text{(5)} \qquad u = -\varepsilon^{-2} R^{-1} (B'P + \varepsilon^2 N')x,$$

where

$$R = \begin{pmatrix} R_s & R_{k_s} \\ R_{k_s} & R_k \end{pmatrix}, \quad B = \begin{pmatrix} B_s & 0 \\ 0 & B_k \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ N_{ks} & N_k \end{pmatrix}$$

and P is the uniquely determined symmetric positive definite matrix satisfying the algebraic Riccati equation

$$\text{(6)} \qquad P(A-BR^{-1}N') + (A-BR^{-1}N')'P - \varepsilon^{-2} PBR^{-1}B'P + Q + \varepsilon^2 (M-NR^{-1}N') = 0.$$

Substitution of the formal power series

$$\text{(7ab)} \qquad P = \varepsilon \sum_{j=0}^\infty P^{(j)} \varepsilon^j, \quad P^{(j)} = \begin{pmatrix} P_{sj} & P_{ksj}' \\ P_{ksj} & P_{kj} \end{pmatrix}$$

into (6) yields after equating terms of equal powers of ε the coeficients $P^{(j)}$; we find that

$$\text{(8a)} \qquad P_{s0} B_s \{R_s - R_{k_s}' R_k^{-1} R_{k_s}\} B_s' P_{s0} = Q_s,$$

$$\text{(8bc)} \qquad P_{k_s 0} = P_{k0} = 0,$$

$$\text{(8d)} \qquad P_{k1}\{A_k - B_k R_k^{-1} N_k'\} + \{A_k - B_k R_k^{-1} N_k'\} P_{k1} + -P_{k1} B_k R_k^{-1} B_k' P_{k1} + M_k - N_k R_k^{-1} N_k' = 0.$$

We will assume that (8d) has a unique symmetric positive definite solution. Then we are in the position to prove the following.

THEOREM 3. *The symmetric formal power series solution* (7) *has a positive radius of convergence and is positive definite. Hence it represents an analytic function, defined in a neighborhood of* $\varepsilon = 0$, *being the unique symmetric, positive definite solution of* (6).

Outline of the proof. Substitution of

$$\text{(9)} \qquad P = \varepsilon P^{(0)} + \varepsilon^2 P^{(1)} + \varepsilon^2 Z$$

in (6) yields

$$\text{(10a)} \qquad F(Z;\varepsilon) \equiv ZH(\varepsilon) + H(\varepsilon)'Z - \varepsilon ZBR^{-1}B'Z + \varepsilon G = 0$$

with

$$\text{(10b)} \qquad H(\varepsilon) = \varepsilon(A-BR^{-1}N') - BR^{-1}B'(P^{(0)} + \varepsilon P^{(1)}).$$

It is shown that $\det(H) \neq 0$, which is equivalent with $\det(\partial F/\partial Z) \neq 0$ at $Z = 0$. From theorem 2 we conclude that (7) has a positive radius of convergence. Furthermore, it can be shown that for

$$\text{(11a)} \qquad x'Px = \varepsilon Q_1(x;\varepsilon) + 0(\varepsilon^3 |x|^2)$$

with

$$\text{(11b)} \qquad Q_1 = x_s' P_{s0} x_s + \varepsilon x_s' P_{s1} x_s + 2\varepsilon x_s' P_{ks1}' x_k + \varepsilon x_k' P_{k1} x_k,$$

the following inequality holds

$$\text{(12)} \qquad Q_1(x;\varepsilon) > \delta\varepsilon |x|^2,$$

where δ is some arbitrary small positive number independent of ε. Consequently, P is positive definite and from theorem 1 it follows that the convergent series (7) is a representation of the analytic, unique, positive definite, symmetric solution of (6). □

In the limit for $\varepsilon \to 0$ the optimal solution has the following behavior. (see [3] for the derivation of this result). Initially, it jumps from (x_{s0}, x_{k0}) to the subspace K:

(13a) $(x_s, x_k) =$

$$(0, x_{k0} - B_k R_k^{-1} R_{ks} (R_s - R'_{ks} R_k^{-1} R_{ks})^{-1} B_s^{-1} x_{s0})$$

Then within the subspace K the optimal solution of

(13b) $\dot{x}_k = A_k x_k + B_k u_k$

with initial value (13a) on performance index

(13c) $J = \int_0^\infty x_k' M_k x_k + 2x_k' N_k u_k + u_k' R_k u_k dt$

has to be found, which is equivalent to solving the Riccati equation (8d).

4. CONCLUDING REMARKS

The main result of this paper concerns the limit behavior of solutions of nearly singular optimal control problems (1) with a nontrivial controllability subspace contained in the kernel of Q. For this class of problems the corresponding singular problem, that is (1) with $\varepsilon = 0$, has a family of solutions, each of them composed of an initial pulse and a singular arc. We presented a method for selecting the correct limit solution and proved the validity of the result.

The class of control systems we analyzed has various applications in technology, see [1]. The non-uniqueness of the singular problem brings about that in the large the behavior of the optimal solution within the subspace K follows from $O(\varepsilon^2)$ terms of the performance index. As a consequence of this the solution may vary in an indefinite way within the subspace K as $\varepsilon \to 0$. This fact may for a great deal explain the observed sensitivity of the singular arc on the fysical parameters in a model for optimal heating and cooling by solar energy [5, p.302], which is of the type we dealt with.

5. LITERATURE

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